INTRODUCTION TO ALGEBRA — HANDOUT 1

1. BINARY RELATIONS

Example 1. "... is a son of ...", "... is a friend of ...", "... knows ...", "... likes...", "... is smaller than ...", "... is equal to...", "... is similar to...", "... divides ...", etc.

Definition 1. R is a **binary relation between** sets X and Y if $R \subseteq X \times Y$.

Definition 2. R is a **binary relation on** set X if $R \subseteq X \times X$.

Representations: set, biparite graph, coordinate system, incidence matrix, directed graph.

- **reflexive**: $\langle x, x\rangle \in R$ for all $x \in X$.
- **irreflexive**: $\langle x, x\rangle \notin R$ for all $x \in X$.
- **symmetric**: If $\langle x, y\rangle \in R$ then $\langle y, x\rangle \in R$ for all $x, y \in X$.
- **antisymmetric**: If $\langle x, y\rangle \in R$ then $\langle y, x\rangle \notin R$ for all *distinct* $x, y \in X$.
- **asymmetric**: If $\langle x, y\rangle \in R$ then $\langle y, x\rangle \notin R$ for all $x, y \in X$.
- **transitive**: If $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$ for all $x, y, z \in X$.
- **trichotomous**: If exactly one of $\langle x, y\rangle \in R$, $\langle y, x\rangle \in R$ or $x = y$ holds for all $x, y \in X$.

Exercise 1.

(A) Show that binary relation R is antisymmetric iff (if and only if) $x = y$ whenever $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$.
(B) Show that a binary relation asymmetric iff it is anti-symmetric and irreflexive.
(C) Show that a transitive relation is irreflexive iff it is asymmetric.

Definition 3. The **identity** relation is defined as

$$D_X := \{\langle x, x\rangle : x \in X\}.$$

The **inverse** of binary relation R is defined as

$$R^{-1} := \{\langle y, x\rangle : \langle x, y\rangle \in R\}.$$

The **composition** of binary relations R and S is defined as

$$R \fatsemi S := \{\langle x, z\rangle : \langle x, y\rangle \in R \text{ and } \langle y, z\rangle \in S \text{ for some } y \in X\}.$$

Exercise 2. Show that binary relation R is

(1) irreflexive iff $D_x \cap R = \emptyset$,
(2) reflexive iff $D_x \subseteq R$,
(3) antisymmetric iff $R \cap R^{-1} \subseteq D_x$,
(4) transitive iff $R \fatsemi R \subseteq R$.

Definition 4. Binary relation $\leq$ is a **partial order** if the following holds:

- $x \leq x$ (*reflexive*)
- $x \leq y \leq z \Rightarrow x \leq z$ (*transitive*)

- $x \leq y \leq x \Rightarrow x = y$ (*antisymmetric*)

Exercise 3. How many different partially orders are on a 3 element set?

Example 2. $\langle \{0,1\}, \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}\rangle$, $\langle \mathbb{Z}, \leq\rangle$, $\langle \mathbb{Q}, \leq\rangle$, $\langle \mathbb{R}, \leq\rangle$, $\langle \mathbb{N}, \leq\rangle$, $\langle \mathbb{N}, |\rangle$, $\langle \mathcal{P}(A), \subseteq\rangle$

Example 3. Let $\mathcal{C}$ be the set of (continuous) functions from $\mathbb{R}$ to $\mathbb{R}$ and let

$$f \leq g \overset{def}{\Longleftrightarrow} \forall x\, f(x) \leq g(x)$$

Then $\langle \mathcal{C}, \leq\rangle$ is a partially ordered set.

Definition 5. Binary relation $<$ is a **strictly partial order** iff the following holds:

- $x \not< x$ (*irreflexive*)
- $x < y \Rightarrow y \not< x$ (*asymmetric*)
- $x < y < z \Rightarrow x < z$ (*transitive*)

Exercise 4. Show that strictly partial orders and partial orders are essentially the same (definitionally equivalent). That is the theory of partial orders extended with explicit definition

$$x < y \overset{def}{\Longleftrightarrow} x \leq y \text{ and } x \neq y.$$

and the theory of strict partial orders extended with explicit definition

$$x \leq y \overset{def}{\Longleftrightarrow} x < y \text{ or } x = y.$$

have the same models.

Definition 6. y **covers** x iff $x < y$ but there is no element z such that $x < z < y$.

Hasse diagram

Definition 7. A partial order is called **total order** it is trichotomous, i.e., exactly one of $x < y$, $y < x$ or $x = y$ holds.

Definition 8. z is an **upper bound** of set H if $x \leq z$ for all $x \in H$. z is an **lower bound** of set H if $z \leq x$ for all $x \in H$.

Definition 9. x is **maximal** if there is no element y such that $x < y$. x is **minimal** if there is no element y such that $y < x$.

Definition 10. x is **the greatest element** of a poset if $y \leq x$ for all $y \in X$. x is **the least element** of a poset if $x \leq y$ for all $y \in X$.

Note that the greatest/least element is unique and maximal, but there can be several distinct maximal/minimal elements which in a poset.

Definition 11. A totally ordered subset of a poset is called **chain**. A set of incomparable elements is called **antichain**.

Definition 12. The **height** $h(X)$ of poset $\langle X, \leq\rangle$ is the largest cardinality of a chain. The **width** $w(X)$ of poset $\langle X, \leq\rangle$ is the largest cardinality of an antichain.

Theorem 1 (Dilworth)**.** For every finite poset $\langle X, \leq\rangle$, there is a partition of X into $w(X)$ chains.

Zorn's Lemma: If every chain has an upper bound in a poset, then it contains a maximal element.

Definition 13. A partial order is called **directed** if every pair of elements has an upper bound.

Definition 14. Binary relation $\sim$ is an **equivalence relation** iff the following holds:

- $x \sim x$ (*reflexive*)
- $x \sim y \Rightarrow y \sim x$ (*symmetric*)
- $x \sim y \sim z \Rightarrow x \sim z$ (*transitive*)

Example 4. equality, similarity, congruence, lines being parallel, etc.

Definition 15. The **equivalence class** of a is the set of elements equivalent to a, i.e., $[a] := \{b : a \sim b\}$. Let the set of equivalence classes of $\sim$ be denoted by $X/\sim$.

Example 5. Vectors are equivalence classes of arrows.

Definition 16. $\Pi \subseteq \mathcal{P}(X)$ is a **partition** of X iff the following hold

- $A \neq \emptyset$ for all $A \in \Pi$,
- $A \cap B = \emptyset$ for all distinct $A, B \in \Pi$,
- $\cup\Pi = X$.

Proposition 1. For any equivalence $\sim$ relation on X, the set of equivalence classes $X/\sim$ is a partition of X. For any partition Π on X, the relation being in the same partition is an equivalence relation (whose equivalence classes are exactly the members of the partition Π).

Definition 17. Binary relation $\leftarrow$ is a **preorder** if the following holds:

- $x \leftarrow x$ (*reflexive*)
- $x \leftarrow y \leftarrow z \Rightarrow x \leftarrow z$ (*transitive*)

Proposition 2. Let $\leftarrow$ be a preorder on X. Define $\sim$ as

$$x \sim y \overset{def}{\Longleftrightarrow} x \leftarrow y \text{ and } y \leftarrow x.$$

Then $\sim$ is an equivalence relation on X. Furthermore, if $x \sim x'$, $y \sim y'$ and $x \leftarrow y$, then $x' \leftarrow y'$. $\rightsquigarrow$ *canonical quotient*