INTRODUCTION TO ALGEBRA — HANDOUT 5

Exercise 1. Let L be the set of subspaces of $\mathbb{R}^3$ (i.e., L contains (singleton of) the origin, $\mathbb{R}^3$, and the lines and planes containing the origin. Consider the partial order $\langle L, \subseteq \rangle$. What are the $\wedge$ and $\vee$ operations in this poset? Show that $\langle L, \subseteq \rangle$ is modular but not distributive.

1. Boolean algebras

Example 1. Let X be a set. Partial ordered set $\langle \mathcal{P}(X), \subseteq \rangle$ is a Boolean algebra.

Definition 1. Let $\langle B, \wedge, \vee \rangle$ be a Boolean algebra. A subset I of B is called an **ideal** iff

- $0 \in I$,
- $a, b \in I \Rightarrow a \vee b \in I$, and
- $a \in I$ and $b \leq a \Rightarrow b \in I$.

A subset F of B is called a **filter** iff

- $1 \in F$,
- $a, b \in F \Rightarrow a \wedge b \in F$, and
- $a \in F$ and $a \leq b \Rightarrow b \in F$.

Example 2.

(1) $F = \{1\}$ is a *trivial* filter and $I = \{0\}$ is a *trivial* ideal.
(2) Let $a \in B$. Then $F_a := \{x : x \geq a\}$ is a *principal* filter and $I_a := \{x : x \leq a\}$ is a *principal* ideal.
(3) Let X be an infinite set. The set of all finite subsets of X is a nonprincipal ideal; and the set of all subsets of X having a finite complement is a nonprincipal filter (the Fréchet filter) in $\langle \mathcal{P}(X), \subseteq \rangle$.

Proposition 1. Let $\langle B, \wedge, \vee \rangle$ be a Boolean algebra.

(1) $I \subseteq B$ is an ideal iff $\{\overline{x} : x \in I\}$ is a filter.
(2) $F \subseteq B$ is a filter iff $\{\overline{x} : x \in F\}$ is an ideal.

Definition 2. Let $\langle B, \wedge, \vee \rangle$ be a Boolean algebra. A subset G of B has the **finite intersection property** if $x_1 \wedge x_2 \wedge \ldots \wedge x_n \neq 0$ for all $x_1, x_2, \ldots, x_n \in G$.

Definition 3. A filter F is called **proper** if $F \neq B$.

Note that every proper filter has the finite intersection property.

Proposition 2.

(1) The intersection of any arbitrary set of filters is a filter.
(2) The union of a $\subseteq$-chain of filters is also a filter.
(3) Every set having the finite intersection property can be extended to a proper filter.

Definition 4.
A filter F is called **ultrafilter** iff exactly one of a and $\overline{a}$ belongs to F.
A ideal I is called **maximal ideal** iff is maximal with respect to the property $1 \notin I$.

Proposition 3.

(1) A filter F is an ultrafilter iff F is maximal with respect to the property $0 \notin F$.
(2) An ideal I is a maximal ideal iff I exactly one of a and $\overline{a}$ belongs to I.
(3) A filter F is an ultrafilter iff $0 \notin F$ and for all x, y $x \vee y \in F$ iff $x \in F$ or $y \in F$.
(4) An ideal I is a maximal ideal iff it is a **prime ideal**, i.e., $1 \notin F$ and for all x, y $x \wedge y \in I$ iff $x \in I$ or $y \in I$.

Theorem 1 (Tarski)**.** Every filter can be extended to an ultrafilter.

Proof. Zorn's lemma. □

Consequently, the ultrafilter extension of the Fréchet filter is a nonprincipal ultrafilter in $\langle \mathcal{P}(X), \subseteq \rangle$.

Exercise 2. Prove that all the ultrafilters in a finite Boolean algebra are principal.

Definition 5. Let $\langle B, \wedge, \vee \rangle$ be a Boolean algebra. Let $\mathcal{U}(B)$ be the set of ultrafilters of B. For all $a \in B$, let

$$N_a := \{U \in \mathcal{U}(B) : a \in U\}. \tag{1}$$

Proposition 4. Let $\langle B, \wedge, \vee \rangle$ be a Boolean algebra and $x, y \in B$. Then $N_0 = \emptyset$, $N_1 = \mathcal{U}(B)$, $N_x \cap N_y = N_{x \wedge y}$, $N_x \cup N_y = N_{x \vee y}$, $N_{\overline{x}} = \mathcal{U}(B) \setminus N_x$.

Theorem 2 (Stone)**.** Every Boolean algebra is isomorphic to a Boolean algebra of sets.

Proof. Consider the map $a \mapsto N_a$. □