INTRODUCTION TO ALGEBRA — HANDOUT 12

1. Vector spaces

Definition 1. Let $\langle F, +, \cdot \rangle$ be a field. Let $\langle V, \oplus, \odot \rangle$ is called a vector space over F if $\oplus V \times V \to V$ and $\odot F \times V \to V$ satisfying the following properties

(1) $(v \oplus u) \oplus w = v \oplus (u \oplus w)$,
(2) $v \oplus u = u \oplus v$,
(3) $\exists o \;\; o \oplus v = v$,
(4) $\exists -v \;\; v + (-v) = o$,
(5) $\mu \odot (\nu \odot v) = (\mu \cdot \nu) \odot v$,
(6) $\mu \odot (v \oplus u) = (\mu \odot v) \oplus (\mu \odot u)$,
(7) $(\mu + \nu) \odot v = (\mu \odot v) \oplus (\nu \odot v)$,
(8) $1 \odot v = v$.

Example 1.

(1) $\mathbb{R}^n$; $\mathbb{Q}^n$; $\mathbb{C}^n$; $\mathbb{Z}_2^n$;
(2) $\mathbb{R}^{n \times m}$;
(3) polynomials; polynomials of degree at most d;
(4) functions; continuous functions; differentiable functions;
(5) sequences; convergent sequences;
(6) $\mathbb{C}$ over $\mathbb{R}$; $\mathbb{R}$ over $\mathbb{Q}$.

Definition 2. Let V be a vector space over field F. Vectors $v_1, \ldots, v_n$ are called **linearly independent** if $\lambda_1 \odot v_1 \oplus \ldots \oplus \lambda_n \odot v_n = o$ then $\lambda_1 = \ldots = \lambda_n = 0$. We say that vectors $v_1, \ldots, v_n$ **span** W if for all $w \in W$ there are $\lambda_1, \ldots, \lambda_n \in F$ such that $w = \lambda_1 \odot v_1 \oplus \ldots \oplus \lambda_n \odot v_n$. A **basis** is a set of linearly independent vectors spanning V.

Theorem 1. If $b_1, \ldots, b_n$ and $c_1 \ldots c_m$ are bases of the same vector space, then $n = m$.

Definition 3. The **dimension** of V, in symbols $dim\, V$, is the number of vectors of a base.

Theorem 2. Let V be a vector space over field F. If $dim\, V = n$, then V is isomorphic to F^n.

Definition 4. Let V be a vector space. W is called a **subspace** of V if W is a vector space with the same operations.

Theorem 3. Let V be a vector space over field F. W is a subspace iff the following holds:

(1) $o \in W$;
(2) if $v, u \in W$, then $v \oplus u \in W$;
(3) if $v \in W$ and $\mu \in F$, then $\mu \odot v \in W$.

Exercise 1. Which of the following sets are subspaces of $\mathbb{R}^3$? (Why?)

- $W_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$,
- $W_3 = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2 = y + 2z\}$,
- $W_2 = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y = z\}$,
- $W_4 = \{(x, y, z) \in \mathbb{R}^3 : 2x - 1 = 3y + 2 = z\}$,
- $W_5 = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \in \mathbb{Q}\}$.