INTRODUCTION TO ALGEBRA — HANDOUT 6

1. PROPERTIES OF OPERATIONS

Definition 1. Unary operation U is called **idempotent** if $U(U(x)) = U(x)$ for all x. Unary operation U is called **invertible** if there is another unary operation W such that $U(W(x)) = x$ for all x; and U is called an **involution** if $U(U(x)) = x$ for all x.

Example 1. Grinding of coffee:

- $\langle\{\text{coffee}, \text{ground coffee}\}, \text{Grinding}\rangle$,
- $\langle\{\text{coffee}, 1/2 \text{ ground coffee}, \text{ground coffee}\}, \text{Grinding}\rangle$,
- $\langle\{\text{coffee}, 1/2 \text{ ground coffee}, \text{ground coffee}\}, 0.5\text{-Grinding}\rangle$,
- $\langle[0,1], \lambda\text{-Grinding}\rangle$, where $\lambda\text{-Gringding}(x) = \min(1, x+\lambda)$.

Turning the volume up and down:

- $\langle\mathbb{Z}, U, D\rangle$, where $U(n) = n+1$ and $D(n) = n-1$,
- $\langle\mathbb{Z}\cap[-9,9], U, D\rangle$, where $D(n) = \max(-9, n-1)$ and $U(n) = \min(9, n+1)$.

Rotation of a dial.

Example 2. The absolute value is an idempotent unary operation on $\mathbb{R}$. Taking the multiplicative inverse is an involution unary operation on $\mathbb{R}^+$.

Definition 2. Binary operation $*$ is **associative** if $(a*b)*c = a*(b*c)$, for all a, b, c.

Definition 3. Binary operation $*$ is **commutative** if $a*b = b*a$ for all a, b.

Definition 4. Binary operation $*$ is **idempotent** if $a*a = a$ for all a.

Definition 5. e is **neutral element** of binary operation $*$ if $e*a = a*e = a$ for all a.

Definition 6. a' is called the **inverse** of a if $a'*a = a*a' = e$.

Example 3. Mixing n-liquids, e.g., mixing alcohol and water: $\langle[0,1], m\rangle$, where $m(x,y) = \frac{x+y}{2}$ or $\langle R^+ \times \mathbb{R}^+, M\rangle$, where $M(a,b) = a+b$.

Exercise 1. In two cups, we have equal amounts of tea and coffee. We put some tea from the tea cup into coffee cup, stir it evenly, and then we put back the same amount of mixture into the tea cup. Question: After the mixing, what is more the amount of tea in the coffee cup or the amount of coffee in the tea cup?

Example 4. Genetic crossing of plants:

- $\langle\text{Plants that cross}, c\rangle$, where c is the crossing operation.
- $\langle\text{Plants} \cup \{\emptyset\}, C\rangle$, where C is the crossing operation resulting $\emptyset$ if the two plants cannot be crossed and $C(\emptyset, x) = C(x, \emptyset) = \emptyset$.

Example 5. Concatenation of texts.

Exercise 2. Find more examples of binary operations in every day life and check which properties they have.