

INTRODUCTION TO ALGEBRA — HANDOUT 3

1. GRAPHS

Definition 1. A $\langle V, E \rangle$ is a **graph** iff V is a set (of vertexes) and E (edges) is a symmetric binary relation on V .

Definition 2. A graph is called **k -colorable** if there is a coloring of the vertexes such that no two adjacent vertexes having the same color.

Theorem 1 (Erdős–De Bruijn). A graph is k -colorable iff its every finite subgraph is k -colorable.

2. PARTIAL ORDERS

Theorem 2 (Dilworth, for non necessarily finite graphs). For every finite poset $\langle X, \leq \rangle$, there is a partition of X into $w(X)$ chains.

Proof. Let $\langle X, \leq \rangle$ be a poset. Let graph $\langle X, E \rangle$ be defined as follows: $E(x, y) \iff x \not\leq y \wedge y \not\leq x$, i.e., we connect the incomparable elements of $\langle X, \leq \rangle$. Graph $\langle X, E \rangle$ is n -colorable iff $\langle X, \leq \rangle$ can be partitioned into n chains. Therefore, the statement follows from (the finite version of) Dilworth theorem and Erdős–De Bruijn theorem. \square

Exercise 1. Construct a partial order in which there is an n -element antichain for all natural number n , but it does not contain an infinite antichain.

Theorem 3 (Wolk–Perles). For all infinite cardinal κ there is a partial order P such that $w(P) = \aleph_0$, but every chain partition of P contains at least κ chains.

3. BINARY RELATIONS

Definition 3. Let X and Y be two sets and let R and S be two binary relations on them, respectively. $\langle X, R \rangle$ is called **isomorphic** to $\langle Y, S \rangle$ if there is a bijection $f : X \rightarrow Y$ such that $\langle x, y \rangle \in R \iff \langle f(x), f(y) \rangle \in S$. In this case, f is called an **isomorphism** between $\langle X, R \rangle$ and $\langle Y, S \rangle$.

Example 1. Partial ordered sets $\langle \mathbb{N}, \leq \rangle$ and $\langle \{2^n : n \in \mathbb{N}\}, | \rangle$ are isomorphic, and $f(n) = 2^n$ is an isomorphism between them.

4. LATTICES

Exercise 2. Show that $\langle \mathbb{N}, | \rangle$ is a lattice.

Exercise 3. Draw the hasse diagrams of (up to isomorphism) all the lattices having at most 5 elements.